

# AN INEQUALITY CONCERNING THE GROWTH BOUND OF A DISCRETE EVOLUTION FAMILY ON A COMPLEX BANACH SPACE

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ABSTRACT. We prove that the uniform growth bound  $\omega_0(\mathcal{U})$  of a discrete evolution family  $\mathcal{U}$  of bounded linear operators acting on a complex Banach space  $X$  satisfies the inequality

$$\omega_0(\mathcal{U})c_{\mathcal{U}}(\mathcal{X}) \leq -1;$$

here  $c_{\mathcal{U}}(\mathcal{X})$  is the operator norm of a convolution operator which acts on a certain Banach space  $\mathcal{X}$  of  $X$ -valued sequences.

## 1. NOTATIONS, DEFINITIONS AND STATEMENT

Let  $X$  be a complex Banach space and let  $\mathcal{L}(X)$  be the Banach algebra of all bounded linear operators acting on  $X$ . The norm of  $X$  and the operator norm on  $\mathcal{L}(X)$  are denoted by  $\|\cdot\|$ . We use the classical notations  $\mathbb{Z}_+$  and  $\mathbb{C}$  for the sets of nonnegative integers and of complex scalars, respectively. As is well-known, the space  $l^\infty(\mathbb{Z}_+, X)$  consisting by all bounded  $X$ -valued sequences becomes a Banach space when we endow it with the "sup" norm, i.e.  $\|(f_n)\|_\infty = \sup_{n \in \mathbb{Z}_+} \|f_n\|$ . Let  $l_0^\infty(\mathbb{Z}_+, X)$  be the subspace of  $l^\infty(\mathbb{Z}_+, X)$  which consists of all sequences  $(f_n) \in l^\infty(\mathbb{Z}_+, X)$  with  $f_0 = 0$ . Also consider  $c_0^0(\mathbb{Z}_+, X)$ , the subspace of  $l_0^\infty(\mathbb{Z}_+, X)$  consisting of all sequences  $(f_n)$  having the property that  $\lim_{n \rightarrow \infty} f_n = 0$ . Obviously,  $c_0^0(\mathbb{Z}_+, X)$  and  $l_0^\infty(\mathbb{Z}_+, X)$  are closed subspaces of the Banach space  $l^\infty(\mathbb{Z}_+, X)$ . Now let  $1 \leq p < \infty$ . By  $l_0^p(\mathbb{Z}_+, X)$  we denote the space of all  $X$ -valued sequences  $f = (f_k)_{k \in \mathbb{Z}_+}$  having the property that  $f_0 = 0$  and

$$\|f\|_p := \left( \sum_{k=0}^{\infty} \|f_k\|^p \right)^{\frac{1}{p}} < \infty.$$

Obviously,  $(l_0^p(\mathbb{Z}_+, X), \|\cdot\|_p)$  is a Banach space.

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**Definition 1.1.** A family  $\mathcal{U} := \{U(n, m) : (n, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+, n \geq m\} \subset \mathcal{L}(X)$  is called a *discrete evolution family* if it satisfies the properties:  $U(m, m) = I$  and  $U(m, n) = U(m, p)U(p, n)$  for all nonnegative integers  $m \geq p \geq n$ .

Here  $I$  denotes the identity operator on  $X$ .

Let us denote by  $\Delta$  the set of all pairs of nonnegative integers  $(n, m)$ , so that  $n \geq m$  and let  $\Omega(\mathcal{U})$  be the set of all real numbers  $\omega$  such that

$$(1.1) \quad \sup_{(n, m) \in \Delta} e^{-\omega(n-m)} \|U(n, m)\| := M_\omega < \infty.$$

Throughout the paper we assume that  $\Omega(\mathcal{U})$  is a non-empty set, i.e. the family  $\mathcal{U}$  is exponentially bounded. The uniform growth bound of  $\mathcal{U}$ , denoted by  $\omega_0(\mathcal{U})$ , is the infimum of  $\Omega(\mathcal{U})$ .

A typical example which provides a discrete evolution family is presented next.

**Example 1.** Let  $\mathcal{A} := \{A_n : n \in \mathbb{Z}_+\}$  be a family of bounded linear operators acting on a Banach space  $X$ . The discrete evolution family associated to the family  $\mathcal{A}$  is the two parameters family  $\mathcal{U}_{\mathcal{A}} := \{U_{\mathcal{A}}(m, n) : m \geq n \in \mathbb{Z}_+\} \subset \mathcal{L}(X)$  given by

$$U_{\mathcal{A}}(m, n) := \begin{cases} A_{m-1}A_{m-2} \cdots A_n, & m > n \\ I, & m = n. \end{cases}$$

Obviously, the family  $\mathcal{U}_{\mathcal{A}}$  is an evolution family in the sense of Definition 1.1. Moreover, every evolution family  $\mathcal{U} := \{U(n, m) : (n, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+, n \geq m\}$  comes in this way. It is enough to set  $A_n = U(n+1, n)$  in order to see this.

For any  $X$ -valued sequence  $f = (f_n)_{n \in \mathbb{Z}_+}$  we consider the discrete inhomogeneous Cauchy Problem

$$(1.2) \quad \begin{cases} x_{n+1} = A_n x_n + f_{n+1}, & n = 0, 1, \dots \\ x_0 = 0. \end{cases}$$

Obviously, the solution of (1.2) is the sequences  $(x_n)$ , given by

$$x_n = \sum_{k=0}^n U_{\mathcal{A}}(n, k) f_k, \quad n \in \mathbb{Z}_+.$$

Let  $\mathcal{M} := \{c_0^0(\mathbb{Z}_+, X), l_0^\infty(\mathbb{Z}_+, X), l_0^p(\mathbb{Z}_+, X)\}$ ,  $\mathcal{X} \in \mathcal{M}$  and let  $\mathcal{U} := \{U(n, m) : (n, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+, n \geq m\}$  be an exponentially bounded discrete evolution family.

For each  $j \in \mathbb{Z}_+$  and each sequence  $f = (f_n) \in \mathcal{X}$  consider the linear operator  $\mathcal{T}_{\mathcal{X}}(j)$  given by

$$(\mathcal{T}_{\mathcal{X}}(j)f)(k) := \begin{cases} U(k, k-j)f_{k-j}, & \text{for all } (k, j) \in \Delta \\ 0, & \text{otherwise.} \end{cases}$$

Since the family  $\mathcal{U}$  is exponentially bounded,  $\mathcal{T}_{\mathcal{X}}(j)$  is well defined and acts on  $\mathcal{X}$ . The family  $\mathcal{T}_{\mathcal{X}} := \{\mathcal{T}_{\mathcal{X}}(j)\}_{j \in \mathbb{Z}_+}$  is a discrete semigroup, i.e.  $\mathcal{T}_{\mathcal{X}}(0)$  is the identity operator on  $\mathcal{X}$  and  $\mathcal{T}_{\mathcal{X}}(j+k) = \mathcal{T}_{\mathcal{X}}(j) \circ \mathcal{T}_{\mathcal{X}}(k)$  for all nonnegative integers  $j$  and  $k$ . It is called the evolution semigroup associated to the discrete family  $\mathcal{U}$  on  $\mathcal{X}$ .

Let  $T$  in  $\mathcal{L}(X)$  be a single operator. In the following,  $\rho(T)$  denotes the resolvent set of  $T$ , i.e. the set of all complex scalars  $z$  for which  $zI - T$  has a bounded inverse in  $\mathcal{L}(X)$ . Also  $\sigma(T) := \mathbb{C} \setminus \rho(T)$  denotes the spectrum of the operator  $T$ . As is well-known the spectrum of  $T$  is a compact and non-empty set. The spectral radius of  $T$ , denoted by  $r(T)$ , is defined as  $r(T) := \sup\{|z| : z \in \sigma(T)\}$ . It is well known (Gelfand spectral radius theorem, 1941) that

$$r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}}.$$

Obviously, this yields

$$(1.3) \quad \ln(r(T)) = \lim_{n \rightarrow \infty} \frac{\ln \|T^n\|}{n} = \omega_0(\{U(n, m) := T^{n-m} : (n, m) \in \Delta\}).$$

Since  $\|T^n\| \leq \|T\|^n$ ,  $r(T) \leq \|T\|$  and hence  $\sigma(T)$  is a subset of  $\{z \in \mathbb{C} : |z| \leq \|T\|\}$ . For each  $z \in \rho(T)$ ,  $R(z, T) := (zI - T)^{-1}$  denotes the resolvent operator of  $T$ . It is well-known that for every  $z \in \rho(T)$ , one has

$$(1.4) \quad \|R(z, T)\| \cdot \text{dist}(z, \sigma(T)) \geq 1.$$

In particular, if  $z_n \in \rho(T)$  and  $z_n \rightarrow z \in \sigma(T)$  then  $\|R(z_n, T)\| \rightarrow \infty$  as  $n \rightarrow \infty$ .

The series  $(\sum_{n=0}^{\infty} \frac{T^n}{z^{n+1}})$  is absolutely convergent on  $\{|z| > r(T)\}$  and its sum is given by

$$(1.5) \quad \sum_{n=0}^{\infty} \frac{T^n}{z^{n+1}} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{T}{z}\right)^n = \frac{1}{z} \cdot \left(I - \frac{T}{z}\right)^{-1} = R(z, T),$$

for every  $z \in \mathbb{C}$  with  $|z| > \|T\|$ .

The next Lemma (whose proof is in Section 2) connects the uniform growth bound of an exponentially bounded evolution family  $\mathcal{U}$  and the spectral radius of  $\mathcal{T}_{\mathcal{X}}(1)$ .

**Lemma 1.1.** *Let  $\mathcal{U}$  be an exponentially bounded evolution family as given above,  $\mathcal{X} \in \mathcal{M}$  and let  $\mathcal{T}_{\mathcal{X}}$  be the discrete evolution semigroup associated to  $\mathcal{U}$  on  $\mathcal{X}$ . Then*

$$(1.6) \quad \omega_0(\mathcal{U}) = \ln r(\mathcal{T}_{\mathcal{X}}(1)).$$

The "convolution" operator  $\mathcal{K}_{\mathcal{X}} : D(\mathcal{K}_{\mathcal{X}}) \subset \mathcal{X} \rightarrow \mathcal{X}$ , associated to the discrete family  $\mathcal{U}$ , is defined by

$$(1.7) \quad D(\mathcal{K}_{\mathcal{X}}) := \{f \in \mathcal{X} : U * f \in \mathcal{X}\}$$

where

$$(1.8) \quad (U * f)(k) := \sum_{j=0}^k U(k, j) f_j, \quad f = (f_n) \in \mathcal{X}, k \in \mathbb{Z}_+.$$

**Theorem 1.1.** *Let  $\mathcal{U}, \mathcal{X}, \mathcal{T}_{\mathcal{X}}$  and  $\mathcal{K}_{\mathcal{X}}$  be as above. The following statements are equivalent.*

1. *The family  $\mathcal{U}$  is uniformly exponentially stable, i.e. its uniform growth bound is negative.*
2. *The evolution semigroup  $\mathcal{T}_{\mathcal{X}}$  associated to the family  $\mathcal{U}$  on  $\mathcal{X}$  is uniformly exponentially stable, i.e.  $r(\mathcal{T}_{\mathcal{X}}(1))$  is less than one.*
3. *For each  $f \in \mathcal{X}$ ,  $U * f$  belongs to  $\mathcal{X}$ .*
4. *The linear operator  $f \mapsto \mathcal{K}_{\mathcal{X}}(f)$  is bounded on  $\mathcal{X}$ .*

For further details, counterparts or different versions of the above result we refer the reader to [1],[2],[3],[8] and the references therein.

In the continuous case, results like the previous one are well-known. For further details we refer the reader to [4], [5], [6], [9] and the references therein.

The proof of Theorem 1.1 for  $\mathcal{X} \in \{c_0^0(\mathbb{Z}_+, X), l_0^\infty(\mathbb{Z}_+, X)\}$  is the same as in [3, Thm. 3.4]. We mention that the 4th statement in Theorem 1.1 is not contained in the statement of [3, Thm. 3.4], but its equivalence with the first three statements is established in the proof.

However for the discrete  $l_0^p(\mathbb{Z}_+, X)$ -version of the above theorem we could not find a reference in the literature. The proof of the present version is similar to that given in [3, Thm. 3.4] so we present part of the argument when  $\mathcal{X} = l_0^p(\mathbb{Z}_+, X)$ .

To prove that the first statement implies the third one, let  $N$  and  $\nu$  be two positive constants such that  $\|U(n, m)\| \leq N e^{-\nu(n-m)}$  for every  $(n, m) \in \Delta$  and let  $f \in l_0^p(\mathbb{Z}_+, X)$ . Thus

$$\begin{aligned} \|U * f\|_p^p &\leq N^p \sum_{n=0}^{\infty} e^{-\nu np} \sum_{k=0}^n e^{\nu kp} \|f_k\|_p^p \\ &\leq N^p \sum_{k=0}^{\infty} e^{\nu kp} \|f_k\|_p^p \sum_{n=k}^{\infty} e^{-\nu np} \\ &\leq \frac{N^p e^{\nu p}}{e^{\nu p} - 1} \|f\|_p^p. \end{aligned}$$

For the proof of **3.**  $\Rightarrow$  **4.** we can argue as in the proof of the first step in [3, Thm. 3.3] and we mention the well-known fact that convergence in  $l_0^p(\mathbb{Z}_+, X)$  implies convergence on coordinates.

Now, we prove that the last statement implies the first one. Since  $\mathcal{K}_{\mathcal{X}}$  is bounded, there exists a positive constant  $c_p$  such that

$$(1.9) \quad \|\mathcal{K}_{\mathcal{X}} f\|_p \leq c_p \|f\|_p \text{ for all } f \in l_0^p(\mathbb{Z}_+, X).$$

Let  $j \geq 1$  be an integer and  $x \in X$ . Let  $f = (f_n) \in l_0^p(\mathbb{Z}_+, X)$  with  $f_j = x$  and  $f_k = 0$  whenever  $k$  is different of  $j$ , and set  $U(n, j) = 0$  when  $n < j$ . Thus inequality (1.9) yields

$$\sum_{n=j}^{\infty} \|U(n, j)x\|^p \leq c_p^p \|x\|^p$$

and from the discrete version of the well-known Datko theorem it follows that the family  $\mathcal{U}$  is uniformly exponentially stable; see for example [10, Thm. 3.4].

The connection with the second statement can be made by following the proof of [3, Thm. 3.4]. Finally, we mention that Lemma 3.1 and Theorem 3.5 from [3] remain valid, with the same proof if  $l_0^p(\mathbb{Z}_+, X)$  or  $l_0^\infty(\mathbb{Z}_+, X)$  replaces  $c_{00}(\mathbb{Z}_+, X)$ . Also, we mention that the assumption  $x(0) = 0$  in Lemma 3.1 from [3] is essential. This is the reason why we consider spaces of sequences having first entry equal to 0,

When the family  $\mathcal{U}$  is uniformly exponentially stable, we let

$$c_{\mathcal{U}}(\mathcal{X}) := \|\mathcal{K}_{\mathcal{X}}\|_{\mathcal{L}(\mathcal{X})} = \sup_{\|f\|_{\mathcal{X}} \leq 1} \|U * f\|_{\mathcal{X}}.$$

**Theorem 1.2.** *Let  $\mathcal{X} \in \mathcal{M}$  and let  $\mathcal{U}$  be a uniformly exponentially stable evolution family acting on  $X$ . Then the following three statements hold true.*

- **(i)** *The following inequality occurs:*

$$(1.10) \quad \omega_0(\mathcal{U}) \cdot c_{\mathcal{U}}(\mathcal{X}) \leq -1.$$

- **(ii)** *The resolvent set  $\rho(\mathcal{T}_{\mathcal{X}}(1))$  contains the set*

$$\pi := \left\{ |z| > 1 - \frac{1}{c_{\mathcal{U}}(\mathcal{X})} \right\}.$$

- **(iii)** *The resolvent operator satisfies the estimate*

$$(1.11) \quad \sup_{|z| \geq 1} \|R(z, \mathcal{T}_{\mathcal{X}}(1))\| \leq c_{\mathcal{U}}(\mathcal{X}).$$

If the discrete evolution family  $\mathcal{U}$  satisfies the convolution condition

$$U(n, j) = U(n - j, 0) \text{ for all } (n, j) \in \Delta,$$

then with  $T = U(1, 0)$  we have that  $T^n = U(n, 0)$ . In this case, the convolution operator is defined by

$$(\mathcal{S}_{\mathcal{X}}f)(n) := (T * f)(n) = \sum_{j=0}^n T^{n-j} f_j, \quad f = (f_j) \in \mathcal{X}.$$

Obviously,  $\mathcal{S}_{\mathcal{X}}$  acts on  $\mathcal{X}$  and it is a bounded linear operator on  $\mathcal{X}$  provided that  $r(T) < 1$ . For further details concerning similar results for strongly continuous semigroups see for example [11]. In this particular case, for each pair  $(n, m) \in \Delta$ , we have that  $U(n, m) = T^{n-m}$ . Moreover,  $c_{\mathcal{U}}(\mathcal{X}) = \|\mathcal{S}_{\mathcal{X}}\|$  and  $r(\mathcal{T}_{\mathcal{X}}(1)) = r(T)$ . The above Theorem 1.2 reads as

**Corollary 1.1.** *Let  $T$  be a single operator in  $\mathcal{L}(X)$  such that  $r(T) < 1$  and let  $\mathcal{X} \in \mathcal{M}$ . Then*

$$(1.12) \quad -1 \geq \|\mathcal{S}_{\mathcal{X}}\|_{\mathcal{L}(\mathcal{X})} \ln(r(T)).$$

A natural question to ask is if the inequality (1.12) is sharp. The next example shows that it can be arbitrarily tight.

**Example 2.** *Let  $X = \mathbb{C}$ ,  $T := \gamma \in (0, 1)$  and  $\mathcal{X} = c_0^0(\mathbb{Z}_+, X)$ . Thus  $\|T^n\| = \gamma^n$  and  $r(T) = \|T\| = \gamma < 1$ . Also*

$$\|\mathcal{S}_{\mathcal{X}}\| = \sup_{n \in \mathbb{Z}_+} \sum_{k=0}^n \gamma^k = \frac{1}{1 - \gamma}$$

and the inequality (1.12) becomes

$$\ln(\gamma) \cdot \frac{1}{1 - \gamma} \leq -1.$$

The equality is attained for  $\gamma \rightarrow 1$  (l'Hôpital's rule).

We note that the above Theorem 1.1 does not provide a negative number  $\sigma$  such that  $\omega_0(\mathcal{U})$  is less than  $\sigma$  while our result does this.

As is well known, if  $T \in \mathcal{L}(X)$  and

$$\sum_{n=0}^{\infty} \|T^n x\| < \infty, \quad \forall x \in X$$

then  $r(T) < 1$ ; see [7] for updated results of this type. For comprehensive information on this subject we refer the reader to [12]. In some sense, this result can be improved to

**Corollary 1.2.** *Let  $T \in \mathcal{L}(X)$  such that  $\sum_{n=0}^{\infty} \|T^n\| := u_1(T) < \infty$ . Then for each  $1 \leq p < \infty$ , one has*

$$(1.13) \quad \ln(r(T)) \cdot u_1(T) \leq \|\mathcal{S}_{\ell_0^p(\mathbb{Z}_+, X)}\|_{\mathcal{L}(\ell_0^p(\mathbb{Z}_+, X))} \ln(r(T)) \leq -1.$$

## 2. PROOFS

We start this section with the proof of Lemma 1.1. We already stated that the discrete evolution family  $\mathcal{U}$  is uniformly exponentially stable if and only if  $r(\mathcal{T}_{\mathcal{X}}(1))$  is less than 1. Lemma 1.1 can be derived from this by a simple scaling argument, as was already done; see for example [3, Thm. 3.5]. However, for completeness, we present here a direct proof using only the definitions stated above. Let  $\omega \in \Omega(\mathcal{U})$ ,  $\mathcal{X} \in \mathcal{M}$  and  $f \in \mathcal{X}$ . After an obvious calculation, we get

$$\|\mathcal{T}_{\mathcal{X}}(m)f\|_{\mathcal{X}} \leq M_{\omega}e^{\omega m}\|f\|_{\mathcal{X}}, \quad \forall m \in \mathbb{Z}_+,$$

where  $M_{\omega}$  is defined in (1.1). Therefore,

$$\frac{\ln(\|\mathcal{T}_{\mathcal{X}}(1)^m\|)}{m} \leq \frac{\ln M_{\omega}}{m} + \omega, \quad \forall m \geq 1.$$

Based on (1.3), the previous inequality yields  $\ln(r(\mathcal{T}_{\mathcal{X}}(1))) \leq \omega$ , which produces

$$(2.1) \quad \ln(r(\mathcal{T}_{\mathcal{X}}(1))) \leq \omega_0(\mathcal{U}).$$

In order to establish the reverse inequality in (2.1), let  $j \in \mathbb{Z}_+$ ,  $j \geq 1$ ,  $x \in X$ ,  $x \neq 0$  and set

$$f_k = \begin{cases} x, & \text{if } k = j \\ 0, & \text{otherwise.} \end{cases}$$

Obviously,  $f = (f_k)$  belongs to  $\mathcal{X}$ . Let  $\nu > \omega_0(\mathcal{T}_{\mathcal{X}}) = \ln(r(\mathcal{T}_{\mathcal{X}}(1)))$  (see (1.3)). Thus there exist  $K_{\nu} \geq 1$  such that

$$\|\mathcal{T}_{\mathcal{X}}(j)\|_{\mathcal{L}(\mathcal{X})} \leq K_{\nu}e^{\nu j}, \quad \forall j \in \mathbb{Z}_+.$$

Therefore, for every  $n \in \mathbb{Z}_+$ , one has

$$(2.2) \quad \|U(n+j, j)x\| = \|(\mathcal{T}_{\mathcal{X}}(n)f)(n+j)\| \leq \|\mathcal{T}_{\mathcal{X}}(n)\|_{\mathcal{L}(\mathcal{X})}\|x\| \leq K_{\nu}e^{\nu}\|x\|.$$

On the other hand

$$(2.3) \quad \|U(n+1, 0)x\| \leq \|\mathcal{T}_{\mathcal{X}}(n)\|_{\mathcal{L}(\mathcal{X})}\|U(1, 0)\| \|x\| \leq K_{\nu}e^{\nu}\|U(1, 0)x\|,$$

where the inequality (2.2) was used. From (2.2) and (2.3) we have that  $\omega_0(\mathcal{U}) \leq \ln(r(\mathcal{T}_{\mathcal{X}}(1)))$ , which completes the proof.

*Proof of Theorem 1.2.*

For every  $z \in \mathbb{C}$ ,  $|z| = 1$ ,  $n \in \mathbb{Z}_+$  and  $f \in \mathcal{X}$ , one has

$$\begin{aligned} [R(z, \mathcal{T}_{\mathcal{X}}(1))f](n) &= \sum_{k=0}^{\infty} \frac{(\mathcal{T}_{\mathcal{X}}(k)f)(n)}{z^{k+1}} \\ &= \sum_{k=0}^n \frac{U(n, n-k)f_{n-k}}{z^{k+1}} \\ &= \frac{1}{z^{n+1}} \sum_{j=0}^n U(n, j)(z^j f_j) \\ &= \frac{1}{z^{n+1}} (U * g)(n), \end{aligned}$$

where  $g_j := z^j f_j$  for all  $j \in \mathbb{Z}_+$  and  $g = (g_j)$ . Clearly,  $f \in \mathcal{X}$  if and only if  $g \in \mathcal{X}$  and, in addition  $\|g\|_{\mathcal{X}} = \|f\|_{\mathcal{X}}$ . Hence

$$(2.4) \quad \|R(z, \mathcal{T}_{\mathcal{X}}(1))f\|_{\mathcal{X}} = \|U * g\|_{\mathcal{X}} \leq \|\mathcal{K}_{\mathcal{X}}\|_{\mathcal{L}(\mathcal{X})} \|g\|_{\mathcal{X}} = c_{\mathcal{U}}(\mathcal{X}) \|f\|_{\mathcal{X}},$$

i.e.

$$(2.5) \quad \|R(z, \mathcal{T}_{\mathcal{X}}(1))\| \leq c_{\mathcal{U}}(\mathcal{X}), \quad \forall z \in \mathbb{C}, |z| = 1.$$

Thus for any  $z \in \mathbb{C}$  with  $|z| = 1$  we have that

$$(2.6) \quad c_{\mathcal{U}}(\mathcal{X}) \geq \|R(z, \mathcal{T}_{\mathcal{X}}(1))\| \geq \frac{1}{1 - r(\mathcal{T}_{\mathcal{X}}(1))},$$

where different counterparts of the result in [3, Thm. 3.5] and (1.4) with  $\mathcal{T}_{\mathcal{X}}(1)$  instead of  $T$ , was used.

Now, (1.10) is a consequence of the elementary inequality

$$(2.7) \quad \frac{1}{1-r} \geq \frac{-1}{\ln(r)}, \quad r \in (0, 1).$$

Thus statement **(i)** is settled.

On the other hand, (2.6) can be written in the form

$$(2.8) \quad r(\mathcal{T}_{\mathcal{X}}(1)) \leq 1 - \frac{1}{c_{\mathcal{U}}(\mathcal{X})}$$

which readily yields **(ii)**, as well.

Since the Neuman series expansion of the resolvent shows that

$$\|R(z, \mathcal{T}_{\mathcal{X}}(1))\| \rightarrow 0 \text{ as } |z| \rightarrow \infty,$$

assertion **(iii)** follows from (2.5) and from the Phragmén-Lindelöf theorem.

**Remark 2.1.** We used [3, Thm. 3.5] and its counterparts in the proof of Theorem 1.2, although it is not needed at all to derive (2.6). Indeed, let us choose  $\lambda \in \sigma(\mathcal{T}_{\mathcal{X}}(1))$  with  $|\lambda| = r(\mathcal{T}_{\mathcal{X}}(1))$  and a complex number  $z$  with  $|z| = 1$  and  $\arg(z) = \arg(\lambda)$ . Then one readily obtains



(2.9)

$$c_u(\mathcal{X}) \geq \|R(z, \mathcal{T}_{\mathcal{X}}(1))\| \geq \frac{1}{\text{dist}(z, \sigma(\mathcal{T}_{\mathcal{X}}(1)))} = \frac{1}{|\lambda - z|} = \frac{1}{1 - r(\mathcal{T}_{\mathcal{X}}(1))}.$$

We thank the referee who brought our attention concerning this important fact.

*Proof of Corollary 1.2.* We divide the proof into two parts by considering the cases  $p = 1$  and  $p > 1$  separately. Let  $f = (f_k) \in l_0^1(\mathbb{Z}_+, X)$ . Then

$$\begin{aligned} \|T * f\|_1 &= \sum_{n=0}^{\infty} \left\| \sum_{k=0}^n T^{n-k} f_k \right\| \\ (2.10) \quad &\leq \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} 1_{\{0,1,\dots,n\}}(k) \|T^{n-k}\| \|f_k\| \\ &= \sum_{k=0}^{\infty} \|f_k\| \sum_{n=k}^{\infty} \|T^{n-k}\| \\ &= u_1(T) \cdot \|f\|_1, \end{aligned}$$

which yields

$$\|\mathcal{S}_{l_0^1(\mathbb{Z}_+, X)}\|_{\mathcal{L}(l_0^1(\mathbb{Z}_+, X))} \leq u_1(T).$$

As is usual,  $1_B$  denotes the characteristic function of the set  $B \subset \mathbb{Z}_+$ .

Multiplying the above inequality by  $\ln(r(T))$  and using Corollary 1.1 (with  $l_0^1(\mathbb{Z}_+, X)$  instead of  $\mathcal{X}$ ) we get

$$-1 \geq \|\mathcal{S}_{l_0^1(\mathbb{Z}_+, X)}\|_{\mathcal{L}(l_0^1(\mathbb{Z}_+, X))} \ln(r(T)) \geq u_1(T) \ln(r(T)).$$

When  $p \in (1, \infty)$ , let  $f = (f_k) \in l_0^p(\mathbb{Z}_+, X)$  and  $h = (h_k) \in l_0^q(\mathbb{Z}_+, X^*)$ , with  $\frac{1}{p} + \frac{1}{q} = 1$ . Then

$$\begin{aligned} (2.11) \quad \left\| \sum_{k=0}^{\infty} h_k(T * f)(k) \right\| &\leq \sum_{k=0}^{\infty} \|h_k\| \|(T * f)(k)\| \\ &\leq \sum_{k=0}^{\infty} \|h_k\| \sum_{j=0}^k \|T^j\| \|f_{k-j}\| \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \|h_k\| 1_{\{0,\dots,k\}}(j) \|T^j\| \|f_{k-j}\| \\ &= \sum_{j=0}^{\infty} \|T^j\| \sum_{k=j}^{\infty} \|h_k\| \|f_{k-j}\|. \end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned} (2.12) \quad \left\| \sum_{k=0}^{\infty} h_k(T * f)(k) \right\| &\leq \sum_{j=0}^{\infty} \|T^j\| \left( \sum_{k=j}^{\infty} \|h_k\|^q \right)^{1/q} \|f\|_p \\ &\leq \|h\|_q \|f\|_p \sum_{j=0}^{\infty} \|T^j\| \\ &= \|h\|_q \|f\|_p u_1(T). \end{aligned}$$

Thus

$$(2.13) \quad \|T * f\|_p = \sup_{\|h\|_q \leq 1} \left\| \sum_{k=0}^{\infty} h_k(T * f)(k) \right\| \leq \|f\|_p u_1(T),$$

and hence  $u_1(T) \geq \|f \mapsto T * f\|_{\mathcal{L}(l_0^p(\mathbb{Z}_+, X))}$ . The assertion follows by using Corollary 1.1 and taking into account that  $\ln(r(T))$  is negative.

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